On Designing of Leader-Follower Impedance Consensus Controllers for Lagrangian Multi-Agent Systems

Abbas Tariverdi[1], Iman Sharifi[2] , Heidar Ali Talebi[3] and Masoud Shafiee[4]

***Abstract*— In this study, an impedance consensus algorithm is presented for leader-following Lagrangian Multi-Agent Systems (MASs) with directed communication topology in the Input-to-State stability (ISS) sense. A general class of nonlinear impedance surfaces based on damped Lagrangian systems are introduced in which the followers are shown to achieve the impedance consensus in finite time, which means the dynamic behavior of the followers reach the desired impedance surface provided that a directed spanning tree exists in the communication topology. This surface is the interactive behaviors of the agents with their contacting environments. Finally, it is proved that the proposed method solves the position and velocity consensus in the leader-follower MASs on desired impedance surface with ISS sense through sliding mode control method.**

I. INTRODUCTION

The variety aspects of control designing problem for networks of multi-agent systems has studied precipitously. Consensus is achieved when all agents in the group reach an agreement. Consensus algorithms are interaction rules that specify the information exchange between an agent and all its neighbors on the networks [1]. The consensus problem of MASs can be ordered into two classification according to whether or not there is a leader in the MASs network. In the case when a leader exists, the consensus problem is called leader-following and if there is not leader in the group, it is called leaderless. In leaderless consensus problem the states of all agents reach an agreement on a common value, which depends on the initial states of the agents. Despite this, in leader-following case all followers states converge to leader states. Finite time ISS of a dynamical system involves trajectories that converge to desirable error band in finite time and remain there for future times. Sufficient conditions for finite time ISS were studied in [2]. Studies have mainly shown consensus of single integrator dynamic, e.g. [1], [3]–[6], and for finite time nonlinear first order consensus include [5], [7]–[10]. Second order MASs consensus were studied in [11]–[14].Forthermore, a consensus problem for finite time nonlinear second order MASs was also investigated in [15]–[18]. In [18], the authors proposed a decentralised finite time consensus controller for the undirected MASs network, that is, the proposed algorithm couldn't handle the MASs with directed communication topology.

In comparison to [18], proposed consensus control laws in [15]–[17] can be applied to directed communication topology. However, in [17] the inherent nonlinear dynamic bound is used in the controller. Therefore, these upper bounds should be known before designing the controller. Although the second order system dynamics are considered to be nonlinear in [16], the dynamic terms are supposed to be a priori known, so, the feedback linearizion is applied. Moreover, in [15] the sliding surface is linear and the user force is ignored. So, it can not be used in the interaction port with the environment. To summarize, the proposed algorithm in this paper can be applied to Lagrangian MASs with directed communication topology which interact with their environments. This could be obtained by introducing a general class of nonlinear impedance surfaces based on damped Lagrangian systems.

In this article, a method for impedance consensus of multi-agent systems is proposed. The impedance consensus is the finite time regulation of the dynamic behavior of the robot in the interaction port with its contacting environment. The sliding mode controller is consisted of two phases, the reaching phase and the sliding phase. In the reaching phase the controller tries to regulates the system on the sliding surface. Consequently, the controller forces the system to remain on the surface which is called the sliding phase. In this paper, the related error of reaching phase converge to zero in finite-time while the sliding phase error is ultimately bounded.

Hence, by presenting the desired impedance surface, we investigate the minimization of the robot dynamic behavior with the target impedance behavior using the sliding mode controller. Afterwards, the consensus of the followers' motions on the leader's motion is established in the ISS sense.

II. MATHEMATICAL BACKGROUND AND PRELIMINARIES

let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be a weighted digraph (or directed graph) of order n with the set of nodes $\mathscr{V} = \{V_1, ..., V_n\}$, set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ and a weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency element a_{ij}. An edge of $\mathscr{G}$ is denoted by $e_{ij}(V_i, V_j)$. The i'th agent can obtain information from agent j if and only if $(i, j) \in \mathscr{E}$ i.e. $e_{ij} \in \mathscr{E} \iff a_{ji}$. Moreover, we assume if $(i, j) \notin \mathscr{E}$ or $i = j$ then $a_{ij} = 0$ that is the agent has no self loops. The set of neighbors of node V_i is denoted by $\mathscr{N} = \{V_j \in \mathscr{V} : (V_j, V_i) \in \mathscr{E}\}$. A directed path is a sequence of edge in a directed graph of the form $(i, j), (j, k)$ and etc. A directed graph is strongly connected if there is a

[1]Abbas Tariverdi is graduate student of Department of Electrical Engineering, Amirkabir University of Technology, 424, Hafez, Tehran, Iran tariverdi.abbas@gmail.com

[2]Iman Sharifi is a Phd student in Department of Electrical Engineering, Amirkabir University of Technology, 424, Hafez, Tehran, Iran imansharifi@aut.ac.ir

[3]H.A. Talebi is with the Department of Electrical Engineering, Amirkabir University of Technology, 424, Hafez, Tehran, Iran alit@aut.ac.ir

[4]Masoud Shafiee is with the Department of Electrical Engineering, Amirkabir University of Technology, 424, Hafez, Tehran, Iran mshafiee@aut.ac.ir

directed path from every node to every other node. A directed path is complete if there is an edge from every node to every other node. A directed tree is a directed graph in which every node has exactly one parent except for one node, called the root, which has no parent and which has directed path to all other nodes. Note that a directed tree has no cycle because every edge is oriented away from the root. Directed graph $\mathscr{G}$ has a directed spanning tree if and only if $\mathscr{G}$ has at least one node with directed paths to all others nodes. In directed graphs, the existence of a directed spanning tree is a weaker condition the being strongly connected. Above Preliminaries are borrowed from [19].

If $\mathscr{G}$ be a strongly connected graph and L denotes its directed Laplacian matrix then algebraic and geometric multiplicity of the zero eigenvalue of L is equal to one [16].

Proposition 1: [20] Let L be a non-symmetric (symmetric) Laplacian matrix associated with the directed(undirected) graph $\mathscr{G}$ of order N then L has at least on zero eigenvalue and all its nonzero eigenvalue have positive real parts(are positive). Furthermore, L has a simple zero eigenvalue if and only if $\mathscr{G}$ has a directed spanning tree (is connected). In addition, there exists a nonnegative vector $P \in \mathbb{R}^N$ satisfying $P^T L = 0$ and $\mathbf{1}_n^T P = 1$.

Proposition 2: [21] Let $\Psi_1(x)$ and $\Psi_2(x)$ be any two real integrable functions in $[a,b]$, then Schwarz's inequality is given by:

$$|\langle \Psi_1 | \Psi_2 \rangle|^2 \leq \langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle$$

Schwarz's inequality is also called the Cauchy-Schwarz inequality in some texts.

Lemma 1 ([22]): The nonlinear dynamical system $\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0 \quad t \geq 0$ is input to state stable if and only if there exist a continuously differentiable radially unbounded, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ and continuous function $\gamma_1, \gamma_2 \in \mathscr{K}$ such that for every $u \in \mathbb{R}^m$, $V'(x) F(x,u) \leq -\gamma_1(\|x\|), \quad \|x\| \geq \gamma_2(\|u\|)$

Lemma 2 ([23]): Suppose there exists a continuous function $V : \mathscr{D} \to \mathbb{R}$ such that the following conditions hold: (i) $V$is positive definite. (ii) There exist real numbers $c > 0$ and $\alpha \in (0,1)$ and an open neighborhood $\mathscr{V} \subseteq \mathscr{D}$ of the origin such that:

$$\dot{V}(x) + c(V(x))^{\alpha} \leq 0, \;\; x \in \mathscr{V} \backslash \{0\}$$

Then the origin is a finite-time-stable equilibrium of $\dot{x}(t) = f(x(t))$. Moreover, if $\mathscr{N}$ is an open neighborhood of the origin and T is the settling-time function, then

$$T(x) \leq \frac{1}{c(1-\alpha)} V(x)^{1-\alpha}, \;\; x \in \mathscr{N}$$

III. Impedance consensus for Lagrangian systems

Euler-lagrange equation on a Riemannian manifold in task space are obtained for each agents as follow:

$$\begin{array}{l} M(x_i)\ddot{x}_i + \dot{x}_i^T \Gamma(x_i,\dot{x}_i)\dot{x}_i + DV_p(x_i) = f_i + f_{ei} \\ i = 1,2,...,N \\ x_i, \dot{x}_i \in \mathbb{R}^m \end{array} \tag{1}$$

where i represents number of agents and $DV_p(x_i)$ contains gravity term of a rigid body such that $DV_p(x_i) = \frac{\partial V_p(x_i)}{\partial x_i}$ and elements of the Γ matrix are Christoffel symbol [24], [25]:

$$\Gamma_{yuk} = \frac{1}{2}\left(\frac{\partial M_{yu}}{\partial q_k} + \frac{\partial M_{yk}}{\partial q_u} - \frac{\partial M_{uk}}{\partial q_y}\right)$$

An equivalent form of (1):

$$\begin{array}{l} M(x_i)\ddot{x}_i + C(x_i,\dot{x}_i)\dot{x}_i + DV_p(x_i) = f_i + f_{ei} \\ i = 1,2,...,N \\ x_i, \dot{x}_i \in \mathbb{R}^m \end{array} \tag{2}$$

A. Legendre Transformation

With generalized momentum P that is defined as $p = -\frac{\partial H(x,p)}{\partial x}$. We can rewrite follower agent dynamic using Legendre transformation as a port Hamiltonian system in the first-order differential equation form, such as below:

$$\begin{array}{l} \dot{x}_i = M^{-1}(x_i)\, p_i \\ \dot{p}_i = C\big(x_i, M^{-1}(x_i)\, p\big)^T M^{-1}(x_i)\, p - DV_p(x_i) + f_i + f_{ei} \end{array} \tag{3}$$

With introducing Hamiltonian $H(q_i,p_i)$:

$$\begin{array}{l} H(q_i,p_i) = \dot{q}_i^T p_i - L(q_i,\dot{q}_i) = \frac{1}{2} {p_i}^T M^{-1}(q_i)\, p_i + V_p(q_i) \\ x_i(0) = x_{0i} \in \mathbb{R}^m \\ \dot{x}_i(0) = \dot{x}_{0i} \in \mathbb{R}^m \end{array} \tag{4}$$

$f_i \in \mathbb{R}^m$ is a vector of applied generalized force, $f_{ei} \in \mathbb{R}^m$ is a vector of external force/disturbance acting on the end-effector of the robot. We assume that the directed graph G is weakly connected, which means there exist at least one directed path from the leader to any agent and the leader is not affected by any other agent's state and can be controlled only by its own. The leader is labeled as 0 and other agent are labeled as $1,2,...,N$. The diagonal matrix $\mathfrak{A} = diag(b_{01}, b_{02}, ..., b_{0N})$ is used to characterized the communication between followers and leader, if agent i can obtain the information from the leader, b_{0i}, otherwise $b_{0i} = 0$.

Remark 1 ([26]): if there exist a directed spanning tree in the information flow (topology) among all agents, then the matrix $L + \mathfrak{A}$ is full rank.

The leader dynamics are described as :

$$\begin{array}{l} \dot{x}_0 = M_0^{-1}(x_0)\, p_0 \\ \dot{p}_0 = C_0\big(x_0, M_0^{-1}(x_0)\, p\big)^T M_0^{-1}(x_0)\, p - DV_{p0}(x_0) + f_0 + f_{e0} \\ x_0 \in \mathbb{R}^n \\ p_0 \in \mathbb{R}^n \end{array} \tag{5}$$

And the followers dynamics are described as following :

$$\begin{array}{l} \dot{x}_i = M_i^{-1}(x_i)\, p_i \\ \dot{p}_i = C_i\big(x_i, M_i^{-1}(x_i)\, p\big)^T M_i^{-1}(x_i)\, p - DV_{pi}(x_i) + f_i + f_{ei} \end{array} \tag{6}$$

Remark 2: It is assumed that digraph G has directed spanning tree and there exist non-negative constant σ_1, σ_2 such that:

$$\begin{array}{l} \Big\| C_i\big(x_i, M_i^{-1}(x_i)\, p_i\big)^T M_i^{-1}(x_i)\, p_i - DV_{pi}(x_i) \\ \quad - C_0\big(x_0, M_0^{-1}(x_0)\, p_0\big)^T M_0^{-1}(x_0)\, p_0 - DV_{p0}(x_0) \Big\| \leq \\ \qquad \sigma_1 \|x_i - x_0\|_2 + \sigma_2 \|p_i - p_0\|_2 \end{array} \tag{7}$$

The time varying difference related to i'th agent's generalized momentum and position with respect to the its neighbors are described as follow:

$$\begin{array}{l} e_{x_i} = \sum\limits_{j=1}^{N} a_{ij}\,(x_i - x_j) + b_i\,(x_i - x_0) \\ e_{p_i} = \sum\limits_{j=1}^{N} a_{ij}\,(p_i - p_j) + b_i\,(p_i - p_0) \\ i = 1, ..., N \end{array} \tag{8}$$

The equation (7) is a mild condition that can be satisfied by the intended dynamic. Equation (8) and can be rewritten as :

$$\begin{array}{l} E_X = \left[(L + diag\,(b_{01}, ..., b_{0N})) \otimes I_m \right] (x - \mathbf{1} \otimes x_0) \\ E_P = \left[(L + diag\,(b_{01}, ..., b_{0N})) \otimes I_m \right] (p - \mathbf{1} \otimes p_0) \end{array} \tag{9}$$

The derivative of equation (9)

$$\begin{array}{l} \dot{E}_X = diag\left(M_1^{-1}\,(x_1)\,, ..., M_N^{-1}\,(x_N) \right) .E_P \\ \dot{E}_P = (L + diag\,(b_{01}, ..., b_{0N})) \otimes I_m . \\ \left(H + \left(\frac{\partial H_0}{\partial x_0} \right) \otimes 1 + (F + F_E) - (f_0 + f_{e0}) \otimes 1 \right) \end{array} \tag{10}$$

where in:

$$\begin{array}{l} H = \left(-\frac{\partial H_1(x_1, p_1)}{\partial x_1} \right)^T, ..., \left(-\frac{\partial H_N(q_N, p_N)}{\partial x_N} \right)^T \\ H_i\,(x_i, p_i) = \dot{x}_i^T\, p_i - L_i\,(x_i, p_i) = \frac{1}{2} x_i M_i^{-1}\,(x_i)\, p_i + V_{pi}\,(x_i) \end{array} \tag{11}$$

Equations (11) is the i'th agent's Hamiltonian equations.

$$-\frac{\partial H_0}{\partial x_0} = C_0 \left(x_0, M_0^{-1}\,(x_0)\, p_0 \right)^T M_0^{-1}\,(x_0)\, p_0 - DV_{p0}\,(x_0) \tag{12}$$

Equation (12) is the leader Hamiltonian equation. We define $e = [1, ..., 1]^T$, $E_x \triangleq [e_{x_1}{}^T, ..., e_{x_N}{}^T]^T$, $E_p \triangleq [e_{p1}{}^T, ..., e_{p_N}{}^T]^T$, $x \triangleq [x_1{}^T, ..., x_N{}^T]^T$, $v \triangleq [v_1{}^T, ..., v_N{}^T]^T$, $F \triangleq [f_1{}^T, ..., f_N{}^T]^T$, $F_E \triangleq [f_{e_1}{}^T, ..., f_{e_N}{}^T]^T$, $F \triangleq [f_1{}^T, ..., f_N{}^T]^T$ and $F_E \triangleq [f_{e_1}{}^T, ..., f_{e_N}{}^T]^T$. For a vector $Z \triangleq [z_1{}^T, ..., z_N{}^T]^T$ define: $\mathrm{sgn}\,(Z) = [\mathrm{sgn}\left(z_1{}^T\right), ..., \mathrm{sgn}\left(z_1{}^T\right)]$. $\|\mathrm{N}\|_2$ is the norm 2 of N and $\|.\|_{E_q}$ is the norm on $\mathbb{R}^2$ induced by the positive definite matrix $M_r(E_q)$.

Remark 3: The impedance consensus of Euler-Lagrange Nonlinear MASs is equivalent to input to state stability sense of (7).

Next, we will present a general framework for the impedance consensus. Targeted impedance that describe the desired dynamical behavior of the robot to the environment forces(external forces) acting on it. The general form of targeted impedance including mass,damping and stiffness terms such that we can use variety type of physical damping include (Linear viscous damping, Air damping, Coulomb damping, Displacement-squad damping,...) and stiffness include (General nonlinear translational spring with symmetric or asymmetric force,...), respectively instead of damping and stiffness terms. Consider the targeted impedance on configuration space $E \subseteq \mathbb{R}^{N \times m}$ with kinetic energy $K_r(E_x, \dot{E}_x)$, potential energy $V_r(E_x)$, damping $\phi_r(\dot{E}_x, t)$, and forcing F_E. Assume that $K_r(E_x, \dot{E}_x) = \frac{1}{2} \dot{E}_x^T M_r(E_x) \dot{E}_x$ where the mass matrix $M_r(E_x) : \mathbb{R}^{N \times m} \to \mathbb{R}^{N \times m}$ is positive definite for $E_x \subset E$. The trajectories E_x evolve according to the targeted impedance such as:

$$\begin{array}{l} M_r(E_x)\ddot{E}_x + DV_r(E_x) + \phi_r(E_x, E_p, t) = F_E \\ DV_r(E_x) : \mathbb{R}^{N \times m} \to \mathbb{R}^{N \times m} \\ \phi(E_x, E_p, t) : \mathbb{R}^{N \times m} \to \mathbb{R}^{N \times m} \\ E_x\,(t_0) = E_p\,(t_0) = 0 \in \mathbb{R}^{N \times m} \end{array} \tag{13}$$

Then the desired impedance surface is defined as:

$$\begin{array}{l} \sigma = \dot{E}_x + M_r{}^{-1} \int\limits_{t_0}^{t} \phi_r(E_x, E_x, t) dt \\ + M_r{}^{-1} \int\limits_{t_0}^{t} DV_r(E_x) dt - M_r{}^{-1} \int\limits_{t_0}^{t} F_E dt = 0 \end{array} \tag{14}$$

Theorem 1: Under *Remark* 2 and with control protocol (15), the system (10) is input to state stable, then we can conclude that the impedance leader-following consensus can be achieved.

$$\begin{array}{l} \bar{F} = \bar{F}_i + \bar{F}_{ii} + \bar{F}_{iii} \\ \tilde{L} = L + diag(b_{01}, b_{02}, ..., b_{0n}) \\ \bar{F}_i = \tilde{L}^{-1} \otimes I_m \,.\, \left([b_{01}, b_{02}, ..., b_{0n}]^T \otimes (f_0 + f_{e0}) \right) \\ \bar{F}_{ii} = -\tilde{L}^{-1} \otimes I_m \,.\, M^{-1}{}_r \left(\phi_r\,(E_x, E_p) + DV_r\,(E_x) - F_E \right) \\ \bar{F}_{iii} = -\tilde{L}^{-1} \otimes I_m \,.\, sign\,(\sigma)\,(z + c') \\ z = \omega \left(\psi_1 \left\| E_x \right\| + \psi_2 \left\| E_p \right\| \right) \\ \omega = \left\| \tilde{L} \right\| \left\| \tilde{L}^{-1} \right\| \quad , \quad c' > 0 \end{array} \tag{15}$$

Proof:

Consider Lyapanov function as :

$$\begin{array}{l} V = 0.5\ \sigma^T \sigma\ \to \\ \dot{V} = \sigma^T \left[\tilde{L} \otimes I_m . (H + e \otimes H_0 + \bar{F} - e \otimes (f_0 + f_{e0})) \right] \end{array}$$

Applying control laws (15), following equations is obtained:

$$\dot{V} = \sigma^T \left[\tilde{L} \otimes I_m . \left(H + e \otimes H_0 + \tilde{L} \otimes I_m \,.\, \bar{F}_{iii} \right) \right] \tag{16}$$

having the following:

$$\begin{array}{l} \left\| \tilde{L} \otimes I_m \right\| = ... = \left\| \tilde{L} \right\| \\ \left\| H + e \otimes H_0 \right\| \le \psi_1 \left\| E_x - e \otimes x_0 \right\|_2 + \psi_2 \left\| E_p - e \otimes p_0 \right\| \end{array}$$

yields

$$\left\| \tilde{L} \otimes I_m . (H + e \otimes H_0) \right\| \le \omega \left(\psi_1 \left\| E_x \right\| + \psi_2 \left\| E_p \right\| \right) \tag{17}$$

Using equation (16) and (17):

$$\begin{array}{l} \dot{V} \le \|\sigma\| \left\| \tilde{L} \otimes I_m . (H + e \otimes H_0) \right\| + \sigma^T \left(\tilde{L} \otimes I_m \,.\, \bar{F}_{iii} \right) \\ \le \omega\ \|\sigma\| \left(\psi_1 \left\| E_x \right\| + \psi_2 \left\| E_p \right\| \right) + \sigma^T \left(\tilde{L} \otimes I_m \,.\, \bar{F}_{iii} \right) \\ \le \omega\ \|\sigma\| \left(\psi_1 \left\| E_x \right\| + \psi_2 \left\| E_p \right\| \right) - \sigma^T \left\{ sign\,(\sigma)\,(z + c') \right\} \\ \le \omega\ \|\sigma\| \left(\psi_1 \left\| E_x \right\| + \psi_2 \left\| E_p \right\| \right) - \sum\limits_{i=1}^{N} \sum\limits_{j=1}^{m} \left| \sigma_{ij} \right| (z + c') \\ \le \omega\ \|\sigma\| \left(\psi_1 \left\| E_x \right\| + \psi_2 \left\| E_p \right\| \right) - \|\sigma\|\,(z + c') \\ \le -c' \left\| \sigma \right\| \\ \le -0.5\ .\ c'\ .\ V^{0.5} \end{array} \tag{18}$$

■

According to Lemma 1 and equation (18), it is concluded that the system can reach the desired impedance surface in finite time and remain on it for future times.

B. Input to State Stability

It must be shown that position and velocity error is ultimately bounded on the targeted impedance surface. For it's investigation, the approach of [27] is used. Indeed its approach is generalaized for multi-agent systems. Again consider the targeted impedance surface:

$$\begin{array}{l} M_r(E_x)\ddot{E}_x + DV_r(E_x) + \phi_r(E_x,E_p,t) = F_E \\ DV_r(E_x) : \mathbb{R}^{N\times m} \to \mathbb{R}^{N\times m} \\ \phi(E_x,E_p,t) : \mathbb{R}^{N\times m} \to \mathbb{R}^{N\times m} \\ E_x(t_0) = E_p(t_0) = 0 \in \mathbb{R}^{N\times m} \end{array}$$

Symmetric matrix $\Xi(E_x)$ defined by:

$$\begin{array}{l} \Xi_{ij}(E_x) = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial^2 V_r}{\partial E_{x_k}\partial E_{x_i}}(E_x)M_{jk}(E_x) + \right. \\ \left. \frac{\partial^2 V_r}{\partial E_{x_j}\partial E_{x_k}}(E_x)M_{ki}(E_x) - \frac{\partial V_r}{\partial E_{x_k}}(E_x)\frac{\partial M_{ij}}{\partial E_{x_k}}(E_x)\right) \end{array} \tag{19}$$

$\dot{M}$ is skew symmetric (due to the Impedance surface's essence, C matrix is considered to be zero.) and hence

$$\dot{E}_{xi}^T \frac{dM_r}{dt}\dot{E}_{xi} = 0 \quad \forall E_{xi}{}^T \in \mathbb{R}^{N\times m} \tag{20}$$

and since matrix $\Xi(E_x)$ is symmetric thus, one can conclude:

$$\begin{array}{l} \dot{E}_x^T \Xi_{ij}(E_x)\dot{E}_x = \dot{E}_x^T D^2 V_r(E_x) M_r(E_x)\dot{E}_x \\ + DV_r(E_x)^T\left(\frac{dM_r}{dt}\right)\dot{E}_x \end{array} \tag{21}$$

Let $\lambda_i(K_r)$ denotes the i'th maximum eigenvalue of kinetic energy K_r related to i'th agent.

Lemma 3 ([27]): Let P,Q be symmetric with appropriate dimensions, s.t. P is positive definite. Then the eigenvalues of $P^{-1}Q$ are real and

$$\begin{array}{l} \min\limits_{z\neq 0}\frac{z^T Qz}{z^T Pz} = \lambda_2\left(P^{-1}Q\right) \\ \max\limits_{z\neq 0}\frac{z^T Qz}{z^T Pz} = \lambda_1\left(P^{-1}Q\right) \end{array}$$

In our discussion, there exist positive real number $\mu_1^*,\mu_2^*,\gamma_1^*,\gamma_2^*,a^*,b^*$ such that for all $E_x,\dot{E}_x$, and s,

$$\mu_1^*\left\|\dot{E}_x\right\|^2 \le \left\|\dot{E}_x\right\|_{E_x}{}^2 \le \mu_2^*\left\|\dot{E}_x\right\|^2 \tag{22}$$

This inequality says that the inertia is upper and lower bounded.

$$\gamma_1^* V_r(E_x) \le \|DV_r(E_x)\|_{E_x}{}^2 \le \gamma_2^* V_r(E_x) \tag{23}$$

This inequality shows the that the potential function is quadratic-like.

$$V(E_x) \ge 0 \tag{24}$$

$$a^*\left\|\dot{E}_x\right\|^2 \ge \dot{E}_x^T\,\Xi(E_x)\,\dot{E}_x \tag{25}$$

$$b^*\left\|\dot{E}_x\right\|^2 \le \dot{E}_x^T\phi\left(\dot{E}_x,s\right) \tag{26}$$

The inequality (26), emphasizes a fully-damped assumption on ϕ_r (Damping term). It is obvious that the minimums of potential function of a mechanical system with assumptions (22)-(26) are asymptotically stable [25]. The next theorem shows an upper bound on Lyapanov function in the case of bounded input, that for its proof, approach of [27] is used.

Suppose that $\varsigma_{max} = \min\{\sqrt{\frac{2}{\gamma_2^*},\frac{4b^*}{b^{*2}+4a^*}}\}$. $\forall\varsigma$ in $0<\varsigma<\varsigma_{\max}$ define following symmetric positive-definite matrices as below:

$$P_{I,\varsigma} = \begin{bmatrix} 0.5 & -0.5\,\varsigma \\ -0.5\,\varsigma & \frac{1}{\gamma_2^*} \end{bmatrix} \tag{27}$$

$$P_{II,\varsigma} = \begin{bmatrix} 0.5 & 0.5\,\varsigma \\ 0.5\,\varsigma & \frac{1}{\gamma_1^*} \end{bmatrix} \tag{28}$$

$$Q_\varsigma = \begin{bmatrix} b^* - \varsigma a^* & -0.5\,\varsigma b^* \\ -0.5\,\varsigma b^* & \varsigma \end{bmatrix} \tag{29}$$

Furthermore, let:

$$\Omega_{I,\varsigma} = \lambda_2\left(P^{-1}{}_{II,\varsigma}\begin{bmatrix} 0.5 & 0 \\ 0 & \frac{1}{\gamma_2^*} \end{bmatrix}\right) \tag{30}$$

$$\Omega_{II,\varsigma} = \lambda_1\left(P^{-1}{}_{I,\varsigma}\begin{bmatrix} 0.5 & 0 \\ 0 & \frac{1}{\gamma_1^*} \end{bmatrix}\right) \tag{31}$$

and

$$\sigma_\varsigma = \frac{1}{\mu_2^*}\lambda_2\left(P^{-1}{}_{II,\varsigma}Q_\varsigma\right) \tag{32}$$

Theorem 2: Suppose $\|F_E(t)\| \le \hat{F} \quad \forall t \in \mathbb{R}$. Let $K = \min\limits_{0<\varsigma<\varsigma_{\max}} \frac{\Omega_{II,\varsigma}\left(1+\varsigma^2\right)}{\sigma^2{}_\varsigma\,\lambda_2\left(P^{-1}{}_{II,\varsigma}\right)}$. then there are constants d,h,σ with appropriate dimensions such that $\Upsilon(t) \le d\,e^{-\sigma t} + h\,e^{-\frac{\sigma}{2}t} + K\hat{F}^2$ (While the E_{xi} that is the solution of desierd impedance surface exists and is in Q.)

Proof: Fixed ς, s.t. $0<\varsigma<\varsigma_{\max}$,let

$$\Theta\left(E_x,\dot{E}_x\right) = \frac{1}{2}\left\|\dot{E}_x\right\|_{E_x}{}^2 + V_r(E_x) + \varsigma\left\langle \dot{E}_x, DV_r(E_x)\right\rangle_{E_x} \tag{33}$$

where $\langle\cdot,\cdot\rangle_{E_x}$ is the inner product on $\mathbb{R}^n$ induced by the norm $\|\cdot\|_{E_x}$ then the equations (22) and (23) with proposition 2 yields

$$\mu_1^*\left[\left\|\dot{E}_x\right\|\ \|DV_r(E_x)\|\right]P_{I,\varsigma}\begin{bmatrix} \left\|\dot{E}_x\right\| \\ \|DV_r(E_x)\| \end{bmatrix} \le \Theta \tag{34}$$

$$\Theta \le \mu_2^*\left[\left\|\dot{E}_x\right\|\ \|DV_r(E_x)\|\right]P_{II,\varsigma}\begin{bmatrix} \left\|\dot{E}_x\right\| \\ \|DV_r(E_x)\| \end{bmatrix} \tag{35}$$

With its time derivative,

$$\begin{array}{l} \frac{d}{dt}\Theta(E_x(t),E_p(t)) = E_p{}^T M_r(E_x)\ddot{E}_x + \frac{1}{2}E_p{}^T M_r(E_x)E_p \\ + E_p{}^T DV_r(E_x) + \varsigma DV_r(E_x)^T M_r(E_x)\ddot{E}_x \\ + \varsigma DV_r(E_x)^T\frac{d}{dt}M_r(E_x)E_p \\ + \varsigma E_p{}^T D^2 V_r(E_x)M_r(E_x)E_p \end{array}$$

With the equations (13), (20), (21):

$$\begin{array}{l} \frac{d}{dt}\Theta(E_x(t),E_p(t)) = E_p{}^T\phi_r(E_p,t) + E_p{}^T F(t) \\ -\varsigma DV_r(E_x)^T DV_r(E_x) - \varsigma DV_r(E_x)^T\phi_r(E_p,t) \\ -\varsigma DV_r(E_x)^T F(t) \\ +\varsigma E_p{}^T\,\Xi(E_x)E_p \end{array}$$

From (25), (26):

$$\begin{array}{l} \dot{\Theta} \le -\left[\left\|E_p\right\|\ \|DV_r(E_x)\|\right]Q_\varsigma\begin{bmatrix} \left\|E_p\right\| \\ \|DV_r(E_x)\| \end{bmatrix} \\ +\|F(t)\|_\infty\begin{bmatrix} 1 & \varsigma \end{bmatrix}\begin{bmatrix} \left\|E_p\right\| \\ \|DV_r(E_x)\| \end{bmatrix} \end{array} \tag{36}$$

equation (34) and $\frac{d}{dt}\Theta = 2\sqrt{\Theta}\frac{d}{dt}\sqrt{\Theta}$ yeild

$$2\sqrt{\Theta}\frac{d}{dt}\sqrt{\Theta} \le -\sigma_\varsigma\Theta + \|F(t)\|_\infty\sqrt{(1+\varsigma^2)\left(\|E_x\|^2 + \|DV_r(E_x)\|^2\right)} \tag{37}$$

This result in:

$$\frac{d}{dt}\sqrt{\Theta} \le -\frac{1}{2}\sigma_\varsigma\sqrt{\Theta} + \sqrt{\frac{1+\varsigma^2}{\mu_1^*\lambda_2(P_{I,\varsigma})}}\frac{\|F(t)\|_\infty}{2} \tag{38}$$

From Lemma 3

$$\sigma_\varsigma = \frac{1}{\mu_2^*}\min_{z\neq 0}\frac{z^T Q_\varsigma z}{z^T P_{II,\varsigma} z}$$

And

$$\begin{aligned}&\sqrt{\Theta(E_x(t), E_p(t))} \le e^{-\frac{\sigma_\varsigma}{2}t}\left(\sqrt{\Theta_0} - \kappa_\varsigma\hat{F}\right) + \kappa_\varsigma\hat{F}\\ &\Rightarrow \Theta \le e^{-\sigma_\varsigma t}\left(\sqrt{\Theta_0} - \kappa_\varsigma\hat{F}\right)^2\\ &+2e^{-\frac{\sigma_\varsigma}{2}t}\left(\sqrt{\Theta_0} - \kappa_\varsigma\hat{F}\right)\kappa_\varsigma\hat{F} + \kappa^2{}_\varsigma\hat{F}^2\end{aligned} \tag{39}$$

Where $\Theta_0 = \Theta(E_x(0), E_p(0))$ and $\kappa_\varsigma = \frac{1}{\sigma_\varsigma}\sqrt{\frac{(1+\varsigma^2)}{\mu_1^*\lambda_2(P^{-1}{}_{I,\varsigma})}}$
Using Lemma 3 and equations(30),(31)

$$\Omega_{I,\varsigma}\Theta(E_x(t), E_p(t)) \le \Upsilon(t) \le \Omega_{II,\varsigma}\Theta(E_x(t), E_p(t)) \tag{40}$$

Therefore,

$$\begin{aligned}h &= \Omega_{II,\varsigma}\left(\sqrt{\frac{\Upsilon(0)}{\Omega_{I,\varsigma}}} - \kappa_\varsigma\hat{F}\right)^2\\ d &= 2\Omega_{II,\varsigma}\kappa_\varsigma\hat{F}\left(\sqrt{\frac{\Upsilon(0)}{\Omega_{I,\varsigma}}} - \kappa_\varsigma\hat{F}\right)\end{aligned}$$

Then we have:

$$\Upsilon(t) \le he^{-\sigma_\varsigma t} + de^{-\frac{\sigma_\varsigma}{2}t} + \Omega_{II,\varsigma}\kappa^2{}_\varsigma\hat{F}^2 \tag{41}$$

For any ς s.t. $0 < \varsigma < \varsigma_{\max}$ It is valuable to considered that Theorem 2 by itself does not give any sense about $E_x(t)$, but if V_r has a strict global minimum, then $V_r^{-1}(\kappa_\varsigma\hat{F})$ gives an estimate of $E_x(t)$. Thus, Theorem 2 ensures ISS. ■

IV. Simulation

In this section, we consider five Lagrangian MASs with dynamics (5), (6), so, assumption (7) is satisfied. Adjacency matrix of their communication is in the following form (42). Also, this network is described in Fig. 1.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\end{pmatrix} \tag{42}$$

Note that, since the first agent obtains information from the leader, therefore, $L + \mathfrak{A}$ is full rank. To evaluate the effectiveness of the proposed method, some simulations are performed with a 2-DOF planar robot manipulator which its links have unit mass, unit length and the acceleration of gravity is taken to be 9.8. Also, initial positions and velocities of the agents are randomly chosen from the region $[-1.8, 1.8] \times [-1.8, 1.8]$ Consider nonlinear targeted impedance as described below:

$$\begin{aligned}&\dot{E}_x - M^{-1}{}_r\int F_E\,dt + K_{1r}\int E_x\,dt + K_{2r}\int E_x{}^2 sign(E_x)\,dt\\ &+ C_{1r}\int\dot{E}_x\,dt + C_{2r}\int\dot{E}_x^2 sign(\dot{E}_x)\,dt = 0\end{aligned}$$

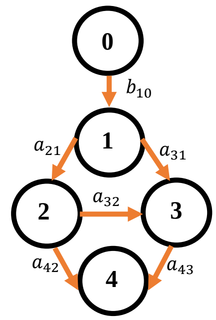

Fig. 1. The network graph topology of the system.

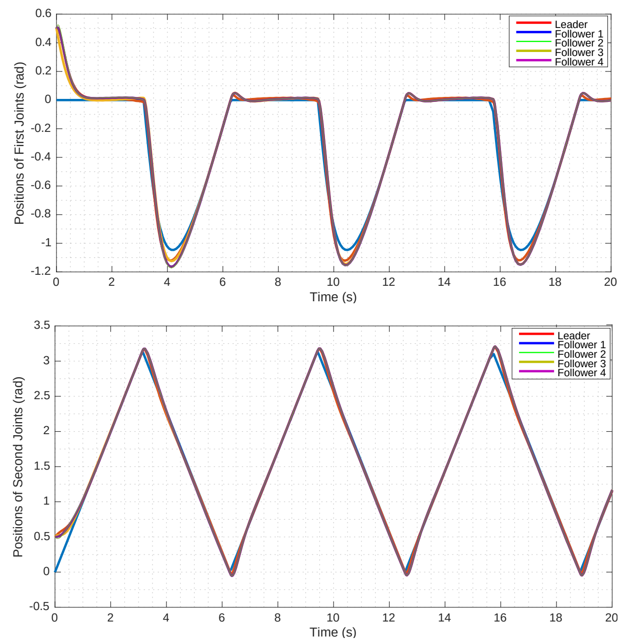

Fig. 2. All the agents' joint positions for q_1 and q_2.

Using consensus protocol (15) and for system (5),(6) motions tracking are shown in Fig 2. Matrices $K_{1r} = 5.5 \otimes I_2$ and $K_{2r} = .5 \otimes I_2$ are desired stiffness parameters, matrices $C_{1r} = 1.7 \otimes I_2$ and $C_{2r} = 1 \otimes I_2$ are desired damping parameters and desired mass term is chose as $M_r = 0.3 \otimes I_2$. The reference trajectory of the leader in Cartesian space is a circle $[x(t), y(t)]^T = \left([0.37\,\cos(t) + 0.5\,,\,0.37\,\sin(t)]^T\right)$. So, to obtain its position in joint space, inverse kinematic is utilized. The results of the system is illustrated in Fig. 3. Moreover, the reference tracking in joint space is depicted in Fig. 3. On the other hand, the performance of linear and nonlinear impedance consensus are compared in Fig.4. All of the simulation results verify the superiority of the proposed method.

V. Conclusion

In this paper, we introduced a nonlinear Lagrangian impedance surface and a distributed leader-following control protocol based on sliding mode method to solve consensus in ISS sense. It is proved that the dynamic behavior of agent can reach on targeted impedance in finite time and, finally,

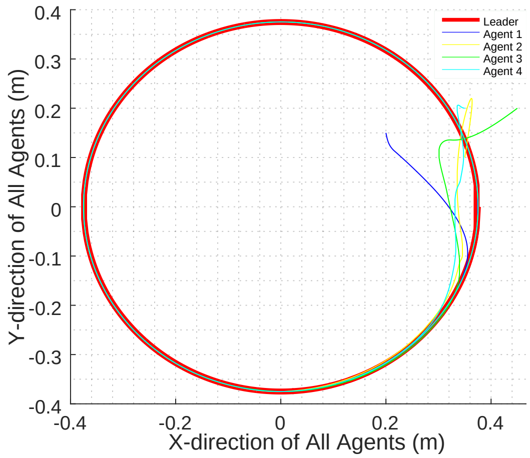

Fig. 3. The circle trajectory of all agents.

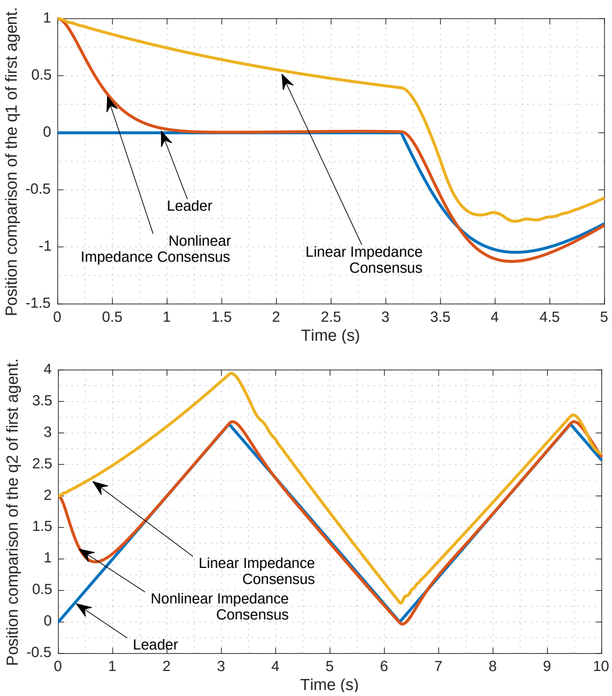

Fig. 4. The first agents q_1 and q_2 is compared in the nonlinear and linear impedance controller.

motions of followers can track the motion of leader. It is investigated that tracking error related to the upper bound of environment/operator forces exerted to each agent. In the term of future work, this method can be applied to the realistic practical situations such as multi-user teleoperation systems.

References

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *Automatic Control, IEEE Transactions on*, vol. 49, no. 9, pp. 1520–1533, 2004.

[2] Y. Hong, Z.-P. Jiang, and G. Feng, "Finite-time input-to-state stability and applications to finite-time control design," *SIAM Journal on Control and Optimization*, vol. 48, no. 7, pp. 4395–4418, 2010.

[3] W. Ren, R. W. Beard, *et al.*, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on automatic control*, vol. 50, no. 5, pp. 655–661, 2005.

[4] W. Ren, "Multi-vehicle consensus with a time-varying reference state," *Systems & Control Letters*, vol. 56, no. 7, pp. 474–483, 2007.

[5] Y. Cao and W. Ren, "Distributed coordinated tracking with reduced interaction via a variable structure approach," *Automatic Control, IEEE Transactions on*, vol. 57, no. 1, pp. 33–48, 2012.

[6] M. Porfiri and D. J. Stilwell, "Consensus seeking over random weighted directed graphs," *Automatic Control, IEEE Transactions on*, vol. 52, no. 9, pp. 1767–1773, 2007.

[7] G. Wen, Y. Yu, Z. Peng, and A. Rahmani, "Distributed finite-time consensus tracking for nonlinear multi-agent systems with a time-varying reference state," *International Journal of Systems Science*, vol. 47, no. 8, pp. 1856–1867, 2016.

[8] Y. Zhang, Y. Yang, Y. Zhao, and G. Wen, "Distributed finite-time tracking control for nonlinear multi-agent systems subject to external disturbances," *International Journal of Control*, vol. 86, no. 1, pp. 29–40, 2013.

[9] Y. Shang, "Finite-time consensus for multi-agent systems with fixed topologies," *International Journal of Systems Science*, vol. 43, no. 3, pp. 499–506, 2012.

[10] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *Automatic Control, IEEE Transactions on*, vol. 55, no. 4, pp. 950–955, 2010.

[11] Y. Hong, G. Chen, and L. Bushnell, "Distributed observers design for leader-following control of multi-agent networks," *Automatica*, vol. 44, no. 3, pp. 846–850, 2008.

[12] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Physica A: Statistical Mechanics and its Applications*, vol. 374, no. 2, pp. 853–863, 2007.

[13] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 40, no. 3, pp. 881–891, 2010.

[14] W. Yu, G. Chen, and M. Cao, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, no. 6, pp. 1089–1095, 2010.

[15] C.-E. Ren and C. Chen, "Sliding mode leader-following consensus controllers for second-order non-linear multi-agent systems," *Control Theory & Applications, IET*, vol. 9, no. 10, pp. 1544–1552, 2015.

[16] M. Ghasemi and S. G. Nersesov, "Finite-time coordination in multiagent systems using sliding mode control approach," *Automatica*, vol. 50, no. 4, pp. 1209–1216, 2014.

[17] H. Li, X. Liao, and G. Chen, "Leader-following finite-time consensus in second-order multi-agent networks with nonlinear dynamics," *International Journal of Control, Automation and Systems*, vol. 11, no. 2, pp. 422–426, 2013.

[18] A.-M. Zou, K. D. Kumar, and Z.-G. Hou, "Distributed consensus control for multi-agent systems using terminal sliding mode and chebyshev neural networks," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 3, pp. 334–357, 2013.

[19] W. Ren and Y. Cao, *Distributed coordination of multi-agent networks: emergent problems, models, and issues*. Springer Science & Business Media, 2010.

[20] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. Springer, 2008.

[21] H. A. Schwarz, *Über ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung*. Springer, 1890.

[22] W. M. Haddad and V. Chellaboina, *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton University Press, 2008.

[23] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[24] J.-J. E. Slotine, W. Li, *et al.*, *Applied nonlinear control*, vol. 199. Prentice-Hall Englewood Cliffs, NJ, 1991.

[25] M. W. Spong, *Robot Dynamics and Control*. New York, NY, USA: John Wiley & Sons, Inc., 1st ed., 1989.

[26] S. Khoo, L. Xie, and Z. Man, "Robust finite-time consensus tracking algorithm for multirobot systems," *Mechatronics, IEEE/ASME Transactions on*, vol. 14, no. 2, pp. 219–228, 2009.

[27] S. Y. Chang, C. R. Carlson, and J. C. Gerdes, "A lyapunov function approach to energy based model reduction," 2001.