Finite-Time Input to State Stability of Discontinuous Dynamical Systems: A Case Study of Networked Euler-Lagrange Systems

Abbas Tariverdi[1†], Heidar Ali Talebi[1,2] and Mahmood Shabankhah [3]

[1]Department of Electrical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran
(Tel: +98-912-744-2363; E-mail: tariverdi.abbas@gmail.com)

[2]Department of Electrical and Computer Engineering, University of Western Ontario, London, Ontario, Canada
(Tel: +98-21-6454-3340 ; E-mail: alit@aut.ac.ir)

[3] School of Engineering Science, College of Engineering, University of Tehran, Tehran, Iran
(E-mail: shabankhah@ut.ac.ir)

Abstract: In this study, Finite-Time Input-to-State Stability (FTISS) is considered for discontinuous dynamical systems based on extended Filippov's solutions and non-smooth Lyapunov functions. The convergence of these systems is analyzed using non-smooth and set-valued analysis, thereby proving the existence of a finite-time input-to-state Lyapunov function for the intended systems implies FTISS of these systems. As an illustration, the results of the theorem are applied to study finite-time input-to-state stability of Multi-Agent Systems (MASs) with Euler-Lagrange dynamics.

Keywords: Finite-Time Stability, Multi-Agent Systems, Filippov's solutions, non-smooth Lyapunov function, Euler-Lagrange dynamics.

1. INTRODUCTION

Analysis of Discontinuous Dynamical Systems (DDSs) emerges in a substantial number of practical and theoretical studies, such as control problems of a robotic manipulator with environmental contacts and non-smooth mechanics.

It is necessary to consider either time-dependent or discontinuous feedbacks or control inputs for the stabilization of many control problems. Sliding mode and variable structure control methods try to stabilize systems with discontinuous feedbacks and control inputs, respectively. Also, in other control methods such as optimal and adaptive control, discontinuous switching algorithms are employed to achieve optimal trajectories, enhance the robustness, and guarantee boundedness of the estimated variables [1].

In non-smooth mechanics, Coulomb friction (as a force related to the direction of slippage between two dry surfaces in contact with each other) and contact interactions with the environment imply not only velocity jumps but also force discontinuities in the motion of rigid bodies. Moreover, discontinuities can also be deliberately designed to attain regulation and stabilization [1–3].

In contrast to differential analysis, non-smooth analysis deals with functions that are not necessarily differentiable. The prevalence of non-differentiable phenomena in real-world applications necessitates such analysis. Non-smooth analysis has proved to be an essential tool in domains ranging from functional analysis to optimal control theory [4].

A function (or single-valued map) maps a point of its domain to a point in its range. However, a set-valued map maps a point of its domain into a set. Set-valued functions arise naturally in applications where there are some uncertainties as to the exact value of some parameters or the behavior of the system under consideration (e.g., disturbances, modeling uncertainties, etc.) [1, 5].

Input-to-State Stability (ISS) property of a system implies that the states remain bounded whenever the inputs are bounded. Another key aspect of ISS is that the states can reach their equilibrium when the inputs tend to zero [6–8]. ISS of dynamical systems have been well studied in the literature [6, 9–13]. In [9–13] the authors consider the ISS of nonlinear systems rather than linear cases. However, the results obtained in [9] and [10] cannot be applied to continuous-time discontinuous systems since these systems do not satisfy Lipschitz continuous vector field conditions.

Trajectories of finite-time input-to-state stable dynamical systems converge to the desired band in finite-time and remain there for all future times. This motivates the development of Finite-Time Input-to-State Stability (FTISS) theory for continuous-time discontinuous systems by using non-smooth Lyapunov functions as opposed to the work done in [14–18]. The results offered by [19–23] concentrates on stability theory for continuous-time discontinuous systems using non-smooth Lyapunov functions.

Proposing an FTISS theorem using the concept of generalized Filippov's solution and non-smooth FTISS Lyapunov functions with a relaxed condition for DDSs, and also extending the line of research opened up in [24–29] are the aims of this paper. Furthermore, the developed FTISS theory is then used to prove finite time input-to-state stability of Multi-Agent Systems (MASs) with Euler-Lagrange dynamics.

Our approach is outlined as follows. First, we introduce an extension of the concept of Filippov's solution, which deals with systems with input/external signals. Here we followed the same line of ideas as in [27]. We then provide a definition of FTISS Lyapunov function similar to [6], [10], [11], [13], [24–26], [29]. It

† Abbas Tariverdi is the presenter of this paper.

should be pointed out that the proposed FTISS Lyapunov function definition enjoys a relaxed condition compared to [18], for example, and the definition covers smooth FTISS Lyapunov functions as well. It is worth noting that our discussion covers FTISS of DDSs in contrast to the papers [27], [28] and [30] where only ISS is proved, and stability is studied regardless of input perturbations. Also, in [2, 3, 31] the authors provide extensions of DDS using Lipschitz continuous Lyapunov functions. We have been able to improve these results in the study of FTISS.

The paper is organized as follows. Mathematical background is briefly discussed in section 2. Section 3 is devoted to the introduced main results on FTISS (Theorem 1). Section 4 outlines the application of the proposed method in Euler-Lagrange MASs in which inherent nonlinear terms assumed to be bounded. Simulations have also been carried out to test the efficiency of our FTISS algorithm in Section 5, and the conclusion is reported in Section 6.

2. MATHEMATICAL FRAMEWORK AND PRELIMINARIES

Here we introduce some necessary notations, concepts, and definitions for systems with discontinuous dynamics. More details can be found in [4], [5], [27], [31–35].

Let $\mathfrak{X}$ be a metric space, and let $\mathtt{S} \subset \mathfrak{X}$. Then *int* S, *co* S, $\bar{co}$ S, *cl* S denote the interior, convex hull, closed convex hull, and closure of S, respectively. Lebesgue measure in R^n is denoted by μ. We write '*a.e.*' whenever a property holds almost everywhere with respect to μ.

A function $f : [a,b] \to R^n$ is called absolutely continuous if it can be expressed in the form $f(t) = f_0 + \int_a^b \mathtt{f}(s)ds$ for some integrable function $\mathtt{f} \in L_1(a,b)$, $L_1(a,b)$ is the set of all locally integrable functions on $[a,b]$; then we have $\dot{f}(t) = \frac{d}{dt}f(t) = \mathtt{f}(t)$ ([4, p.162]). In the scalar case, $f : [a,b] \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_1^n \|f(y_i) - f(x_i)\| < \varepsilon$ whenever $a \le x_1 < y_1 \le x_2 < y_2 \le \ldots \le x_n < y_n \le b$ and $\sum_1^n (y_i - x_i) < \delta$ [1, p. 39]. The collection of all absolutely continuous functions on the interval $[a,b]$ is denoted $AC[a,b]$.

The space of locally absolutely continuous functions on R, i.e. $AC_{loc}(R)$, is defined by $AC_{loc}(R) = \{f : R \to R : f \in AC[a,b], \forall\ a < b\}$. The symbols $\|\cdot\|$ and $\|\cdot\|_P$ in R^n denote the 2-norm and the P-norm induced by the positive-definite matrix P, respectively. For a real-valued differentiable function f, ∂f denotes its generalized gradient. A function $f : R^+ \to R^+$ is said to be positive definite, if $f(0) = 0$ and $f(t) > 0$ for $t > 0$. The class $\mathscr{K}$ consists of all continuous, strictly increasing functions $\psi : R^+ \to R^+$ such that $\psi(0) = 0$. The subclass of functions $\psi \in \mathscr{K}$ which satisfy $\psi(u) \to \infty$ as $u \to \infty$, is denoted by $\mathscr{K}_\infty$. We denote by $\mathscr{KL}$ the set of continuous functions $\beta : R^+ \times R^+ \to R^+$ which, for every fixed $t \in R^+$, the function $\beta(\cdot,t)$ is in $\mathscr{K}$ and, for every fixed $s \in R^+$, the function $\beta(s,\cdot)$ is decreasing and $\beta(s,\cdot) \to 0$ as $s \to \infty$.

A set-valued map $\mathtt{F} : \mathtt{X} \mapsto \mathtt{Y}$ maps every $x \in \mathtt{X}$ to a set $\mathtt{F}(\mathtt{x}) \subset \mathtt{Y}$. $\mathtt{F}(x)$ is called the image of x under F. Suppose that X and Y are subsets of euclidean spaces. Given $x \in \mathtt{X}$ and F as above, we define $\bar{\mathscr{B}}(\mathtt{F}(x),d) = \{y : \mathscr{B}(y,d) \cap \mathtt{F}(x) \neq \emptyset\}$ where $\mathscr{B}(y,d)$ is the open ball with center y and radius d.

A set-valued map $\mathtt{F} : \mathtt{X} \mapsto \mathtt{Y}$ is said to be locally Lipschitz if for every $x_0 \in \mathtt{X}$, there exists a neighborhood $N(x_0) \subset \mathtt{X}$ and a constant $L \geq 0$ such that $\mathtt{F}(x) \subseteq \bar{\mathscr{B}}(\mathtt{F}(x'),\ L\|x - x'\|)$ for every $x, x' \in N(x_0)$. If there exists a constant $L \geq 0$ such that $\mathtt{F}(x) \subseteq \bar{\mathscr{B}}(\mathtt{F}(x'), L\|x - x'\|), \forall x, x' \in X$ then F is called Lipschitz.

The function $f(t)$ is said to be of (differentiability) class C^k if the its derivatives for $k \in \{1,2,\cdots\}$ exist and are continuous (the continuity is implied by differentiability for all the derivatives except for $f^{(k)}(t)$).

Definition 1 ([2]) : Let $V : R^n \to R$ be a locally Lipschitz continuous function. The generalized gradient ∂V of V is defined by $\partial V(x) = co\{\lim \nabla V(x_i) | x_i \to x, x_i \notin \mathscr{S}_V \cup N\}$ where $\mathscr{S}_V$ is the set of Lebesgue measure zero where ∇V does not exist for some arbitrary set of zero measure N.

Lemma 1 ([2, Theorem 3]) : Let $V(x) = \max_{j \in Y} f_j(x)$ where $f_j : R^m \to R$ are C^1 functions and Y is a finite index set and $x : R \to R^m$ be differentiable at t. If $\frac{d}{dt}[V(x(t))]$ exists, then $\frac{d}{dt}[V(x(t))] = \xi^T \dot{x}, \forall \xi \in \partial V(x)$.

Lemma 2 ([3]) : Consider the vector differential equation $\dot{x}(t) = f(x(t))$ where $x(t) = [x_1(t), \ldots, x_n(t)]^T$ and f is not necessarily continuous and belongs to the set-valued map $\mathtt{F} : \mathtt{X} \mapsto \mathtt{Y}$, then $\mathscr{L}_{\mathtt{K}} V = \bigcap_{\xi \in \partial V} \xi^T \mathtt{K}[f](x(t))$ denotes the set-valued Lie derivative of V with respect to the equation $\dot{x}(t) = f(x(t))$.

Lemma 3 ([11]) : Let α be a positive-definite continuous function. Then there exists $\beta(s,t) \in \mathscr{KL}$ with the property that for every locally absolutely continuous function $y : [0,\infty) \to [0,\infty)$, with $y(0) = y_0 \geq 0$, and satisfying $\dot{y}(t) \leq -\alpha(y(t))$, for almost all t, one has $y(t) \leq \beta(y_0,t), \forall t \geq 0$.

3. FTISS FOR DISCONTINUOUS DYNAMICAL SYSTEMS

In this paper, we offer a new approach to combine finite-time control with the concept of ISS. Consider the differential equation with continuous-time discontinuous right-hand side of the form

$$\dot{x} = f(x(t), u(t)). \tag{1}$$

where $x(t) \in R^n$ is the state vector and $u(t) \in R^m$ is the control input at time $t \in R^+$. The vector field f is assumed to be defined *a.e.* and measurable in an open region $\mathtt{Y} \in R^{n+1}$. In addition, for all compact set $\mathtt{D} \in \mathtt{Y}$, an integrable $\mathtt{A}(t)$ exists such that $\|f(x(t),u(t))\| \leq \mathtt{A}(t)$ *a.e.* in D ([2]).

Definition 2 ([27]) : A vector function $x(\cdot)$ is called an extended Filippov's solution of (1) on $[t_0,t_1]$, if $x(\cdot)$ is locally absolutely continuous on $[t_0,t_1]$, for almost all $t \in [t_0,t_1]$, and for every fixed input $u \in L_\infty$, $\dot{x} \in \mathtt{K}[f](x,u)$ where $\mathtt{K}[f](x,u) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(\bar{N})=0} \bar{co} f(\mathscr{B}_\varepsilon(x) \backslash \bar{N}, u)$ where $\bigcap_{\mu(\bar{N})=0}$ denotes the intersection over all sets $\bar{N}$ of Lebesgue measure zero.

Proposition 1: Let $f : R^n \times R^m \to R^n$ be a locally bounded function. Then there exists a set of measure zero $\bar{N}_f \subset R^m$ such that for every $\bar{N} \subset R^m$ with zero measure, and for every fixed input $u \in L_\infty$, we have $\mathtt{K}[f](x,u) = co\{\lim f(x_i,u)|x_i \to x,\ x_i \notin \bar{N}_f \bigcup N\}$.

Proof: This proposition is an extension of part 1 of Theorem 1 in [2]. Here we skip the proof for brevity. ■

Proposition 1 states that $\mathtt{K}[f](x,u)$ is defined as the convex hull of all limit points $\lim_{i\to\infty} f(x_i,u)$ where $x_i \to x(i \to \infty), x_i \notin \bar{N}_f \bigcup N$.

Definition 3 ([18]) : System (1) is called globally finite-time input-to-state stable, if for any initial time $t_0 \geq 0$, initial state $x(t_0) = x_0 \in R^n$, and input $u \in L_\infty$, corresponding Filippov's solution $x(t)$ of the system (1) exists on $[t_0,t)$ and satisfies $\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + \gamma(\sup_{t_0\leq\tau\leq t}\|u(\tau)\|)$, where γ is a $\mathscr{K}$-function and $\beta \in \mathscr{K}\mathscr{L}$ with $\beta(\|x_0\|, t-t_0) \equiv 0$ for $t \geq t_0 + T$ in which T depends continuously on x_0.

Definition 4: A function $V : R^m \to R$ is said to be an FTISS-Lyapunov function for the system (1) if:

i. There exists $\psi_1, \psi_2 \in \mathscr{K}_\infty$ such that: $\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|)$, for every $x \in R^n$.

ii. for $x : R^+ \to R^n$ and $V(x(t))$ is absolutely continuous on $[t_0,\infty)$, there exists $\chi \in \mathscr{K}$ and $\varepsilon > 0$ such that $\{\frac{d}{dt}[V(x(t))] \leq -\varepsilon < 0\}$ *a.e.* on set $\{t : \|x(t)\| \geq \chi(\|u(t)\|)\}$.

Theorem 1: If there exists an FTISS-Lyapunov function in the sense of Definition 4 then the system (1) is globally finite-time input-to-state stable.

Proof: First, We prove by contradiction that there exists $T_0 \geq t_0$ such that $\|x(T_0)\| \leq \chi(\|u(T_0)\|)$. Suppose there exists no such T_0, then $\|x(t)\| \geq \chi(\|u(t)\|)$; $\forall t \in [t_0,\infty)$ and $\dot{V} < -\varepsilon$ *a.e.* on $[t_0,\infty)$, then $\lim_{t\to\infty} V(x(t)) = V(x(0)) + \int_{t_0}^{\infty} \dot{V}(x(t))dt = -\infty$ which contradicts part (i) of Definition 4. It remains to prove that if there exists $T_0 \geq t_0$ such that $x(T_0) \in S = \{\eta : V(\eta) \leq c\}$ with $c = \psi_2(\chi(\|u\|))$ then $x(t) \in S$ for all $t \geq T_0$. Note that if $x(t) \notin S$, then $\psi_2(\|x(t)\|) \geq V(x(t)) \geq \psi_2(\chi(\|u\|))$, which implies that $\|x(t)\| \geq \chi(\|u\|)$. It follows from Definition 4 that $\frac{d}{dt}V(x(t)) \leq -\varepsilon$, if $x(t) \notin S$. Suppose, to get a contradiction, that $x(\tilde{t}) \notin S$ for some $\tilde{t} > T_0$. Since V is continuous, the set S is closed. Hence, there exists $h > 0$, such that $V(x(\tilde{t})) \geq c + h$. Define $t^* = \inf\{t \geq T_0 | V(x(t)) \geq c+h\}$ and $t_* = \sup\{T_0 \leq t \leq t^* | V(x(t)) \leq c\}$. Clearly, $T_0 \leq t_* \leq t$ and $V(x(t_*)) = c$ by the continuity of $V(x(t))$. Also, by the first part of the proof, we understand that the set $\{T_0 \leq t \leq t^* | V(x(t)) \leq c\}$ is not empty.

Since $x(t) \notin S$ for $t \in (t_*, t^*)$, it follows that $\frac{d}{dt}V(x(t)) \leq -\varepsilon$ provided that both $\frac{d}{dt}V(x(t))$ and $\dot{x}(t)$ exist. Since V is absolutely continuous, the mapping $t \to V(x(t))$ is differentiable almost everywhere with respect to t. Hence, $\frac{d}{dt}V(x(t))$ and $\dot{x}(t)$ exist *a.e.*. Therefore,

$$V(x(t^*)) - V(x(t_*)) = \int_{t_*}^{t^*} \frac{d}{dt}V(x(\tau))d\tau \leq \int_{t_*}^{t^*} -\varepsilon d\tau \leq 0.$$

hence, $V(x(t^*)) \leq V(x(t_*)) = c$, which violates the condition $V(x(t^*)) \geq c + h$. This implies that if $x(T_0) \in S$ for some $T_0 \geq t_0$, then $x(t) \in S$ for all $t \geq T_0$.

Now we need to show that the solution $x(t)$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma\big(\sup_{t_0\leq\tau\leq t}\|u(\tau)\|\big).$$

Note that if $t \geq T_0$ then $x(t) \in S$ which gives $V(x(t)) \leq \psi_2(\chi(\|u\|))$. This implies that $\|x(t)\| \leq \gamma(\|u(t)\|)$ which was shown above. We can rewrite this as $V(x(t)) \leq \psi_2 \circ \chi(\|u(t)\|)$ where $\chi = \psi_1^{-1} \circ \psi_2 \circ \chi$. If $t < T_0$, then $x(t) \notin S$, which implies that $\|x(t)\| \geq \chi(\|u(t)\|)$ for all $t \leq T_0$. Consequently, $\frac{d}{dt}V(x(t)) = \nabla V(x(t))f(x(t),u(t)) \leq -\varepsilon < 0$ for almost all $t \leq T_0$.

The last inequality guarantees that $x(t)$ is defined for all $t \geq t_0$. Also,by the comparison principle (lemma (3)), there exists $\tilde{\beta} \in \mathscr{K}\mathscr{L}$ such that $V(x(t)) \leq \tilde{\beta}(V(x_0), t-t_0)$ for $t \leq T_0$. Hence $\|x(t)\| \leq \beta(\|x_0\|, t-t_0)$ for $t \leq T_0$, where $\beta(r,t) = \psi_1^{-1}\tilde{\beta}(\psi_2(r),t)$ is a $\mathscr{K}\mathscr{L}$-function. Combining $\|x(t)\| \leq \gamma(\|u(t)\|)$ for $t \geq T_0$, and $\|x(t)\| \leq \beta(\|x_0\|, t-t_0)$ for $t \leq T_0$, we have $\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + \gamma(\|u(t)\|)$ for $t \geq 0$. Since β and γ are independent of x_0 and u, and from Definition 3, we conclude that the system is globally finite time input-to-state stable. ■

4. ILLUSTRATIVE CASE STUDY

In this section, MASs with Euler-Lagrange dynamics are considered, and consensus problem with a time-varying reference state and directed communication topologies is formulated. Afterward, decentralized sliding mode controllers are designed. In comparison to [36, 37], proposed non-smooth consensus control law can guarantee finite-time consensus for nonlinear second-order MASs as well as it is applicable in directed communication topologies. Finally, a simulation is performed with a 2-DOF planar robot manipulator.

4.1. Graph Theory

In MASs, the information exchange is modeled by a weighted digraph (or directed graph). Suppose $G = (V,E,A)$ with the node set $V = \{V_1, \ldots V_N\}$, set of edges $E \subseteq V \times V$ and a weighted adjacency matrix $A = [a_{ij}]_{N\times N} \in R^{N\times N}$, $a_{ij} \geq 0$. If $a_{ji} > 0$ the i'th agent receives information from agent j that is $e_{ij} = (V_i, V_j) \in E$ and vice versa. $a_{ij} = 0$ if $e_{ij} = (V_i, V_j) \notin E$. Also, there is no self-loop $(i = j)$. The neighbors of node i are defined as $N_i = \{V_j \in V : (V_j, V_i) \in E\}$. A Sequence of edges in a directed graph of the form $e_{ij}, e_{jk}, \ldots$ is called a directed path. Graph G has at least one node with directed paths to all others nodes if and only if digraph G has a directed spanning tree. The Laplacian matrix $L = [l_{ij}]_{N\times N}$ of graph G is defined as $[l_{ij}]_{N\times N} : l_{ij} = -a_{ij},\ \forall i \neq j$, $l_{ii} = \sum_{j\in N_i, j\neq i} a_{ij},\ \forall i,j \in \{1,\ldots,N\}$. Also, $L = D - A$ in which $D = \mathrm{diag}[d_1,\ldots,d_n]$ where $d_i = \sum_{j\in N_i} a_{ij},\ \forall i,j \in \{1,\ldots,N\}$.

4.2. Notation

Consider a vector $\psi = (\psi_1, \ldots \psi_N)$, $\psi_i \in R$ and a matrix Φ, norm p of vector is $\|\psi\|_p := (\sum_{i=1}^n |\psi_i|^p)^{1/p}$ for real number $p \geq 1$ and for matrix Φ, $\|\Phi\|_2 :=$

$\lambda_{\max}(\Phi^*\Phi)^{1/2}$ where $\lambda_{\max}(\cdot)$ denote maximum eigenvalue of matrix Φ. Also, $\sigma_{max}(\Phi)$ and $\sigma_{min}(\Phi)$ denotes maximum and minimum singular values of Φ, respectively. Let $\otimes$ denotes Kronecker product and I_n denotes $n\times n$ identity matrix. We define signum function $sgn(\psi)=[sgn(\psi_1),\cdots,sgn(\psi_N)]$ and diagonal matrix $\text{diag}(\psi)=\text{diag}(\psi_1,\cdots,\psi_N)$.

4.3. Dynamics of MASs and Problem Formulation

The dynamic model of the leader and followers is described in the task space by the following equations

$$\begin{aligned}M_{x0}(q_0)\ddot{x}_0+C_{x0}(q_0,\dot{q}_0)v_0+g_{x0}(q_0)\\ +d_0(q_0,\dot{q}_0,t)=f_0+f_{0e}\\ M_{xi}(q_i)\ddot{x}_i+C_{xi}(q_i,\dot{q}_i)v_i+g_{xi}(q_i)+d_i(q_i,\dot{q}_i,t)=f_i+f_{ie}\end{aligned} \tag{2}$$

where $x_0,\ v_0,\ x_i,\ v_i\in R^n$. Also, $M_{xi}(q_i)$ and $C_{xi}(q_i,\dot{q}_i)$ are $n\times n$ symmetric positive definite inertia matrix and Coriolis centripetal matrix respectively, $g_{xi}(q_i)$ is a $n\times 1$ vector of gravity force, f_i is a $n\times 1$ vector of applied force, f_{ie} is a $n\times 1$ vector of external force acting on the end-effector of the robot and $d_i(q_i,\dot{q}_i,t)$ denotes a generalized non-conservative force, Also, $x_i,\ v_i\in R^n$ denote the position and linear velocity of i'th end-effector in Cartesian space. Following this, the above equation reform such as below

$$\begin{aligned}&\dot{x}_i=v_i\\ &\dot{v}_i(t)=\bar{h}_i(q_i,\dot{q}_i)+\bar{f}_i+\bar{f}_{ie}+\bar{d}_i(x_i,v_i,t),\ i\in\{0,1,\ldots,N\},\end{aligned} \tag{3}$$

where $\bar{f}_i=M_{xi}^{-1}f_i,\ \ \bar{f}_{ie}=M_{xi}^{-1}f_{ie}$, and $\bar{h}_i(q_i,\dot{q}_i)=-M_{xi}^{-1}\Big(C_{xi}(q_i,\dot{q}_i)v_i+g_{xi}(q_i,\dot{q}_i)+d_i(q_i,\dot{q}_i,t)\Big)$.

Assumption 1: For $i\in\{0,1,\ldots,N\}$, $M_{xi}(\cdot)$ is a positive-definite matrix and M_{xi}, C_{xi}, g_{xi} are C^1 functions on states of each system $[x_i,v_i]^T$. In addition, we assume that external force f_{ie} is locally bounded.

Assumption 2: The directed graph G is weakly connected and the desired trajectory $[x_0,v_0]^T$ is C^1 function on $[t_0,\infty)$.

Remark 1: In the dynamics of leader and followers (3), no continuity assumption is made on nonlinear terms $\bar{h}_i(q_i,\dot{q}_i)$, $\bar{h}_0(q_0,\dot{q}_0)$, and $\bar{h}_i(q_i,\dot{q}_i)-\bar{h}_0(q_0,\dot{q}_0)$ so that discontinuous models of friction may be used in generalized non-conservative forces.

Now we define each agent's consensus error as below:

$$\begin{aligned}&e_{xi}=\textstyle\sum_{j=1}^{N}a_{ij}(x_i-x_j)+\alpha_{i0}(x_i-x_0)\\ &e_{vi}=\textstyle\sum_{j=1}^{N}a_{ij}(v_i-v_j)+\alpha_{i0}(v_i-v_0),\ i\in\{0,1,\ldots,N\}.\end{aligned}$$

we rewrite above equations in collective form:

$$\begin{aligned}&\dot{\Lambda}_1=(\tilde{L}\otimes I_m)\big(\dot{X}_N-\dot{X}_r\big)\\ &\dot{\Lambda}_2=(\tilde{L}\otimes I_m)\big(\ddot{X}_N-\ddot{X}_r\big)=(\tilde{L}\otimes I_m)\big(\bar{H}-\bar{H}_0+\bar{F}-\bar{F}_0\big),\end{aligned} \tag{4}$$

where $\tilde{L}=L+\text{diag}(\alpha_{10},\alpha_{20},\ldots,\alpha_{n0})$ such that $\exists\alpha_{i0}\geq 0$, $\forall i\in\{1,\cdots,N\}$ and $\tilde{L}$ is a full rank matrix. Scalars α_{i0}, $\forall i\in\{1,\cdots,N\}$ describe the links between the leader and followers. In addition, $X_N=[x_1^T,\ldots,x_N^T]^T\in R^{n\times N}$, $X_r=1\otimes x_0$, $\bar{H}=[\bar{h}_1^T(q_1,\dot{q}_1),\ldots,\bar{h}_N^T(q_N,\dot{q}_N)]^T$, $\bar{H}_0=1\otimes(\bar{h}_0^T(q_0,\dot{q}_0))$, $\bar{F}=[\bar{f}_1^T+\bar{f}_{1e}^T,\ldots,\bar{f}_N^T+\bar{f}_{Ne}^T]^T$, $\bar{F}_E=[\bar{f}_{1e}^T,\ldots,\bar{f}_{Ne}^T]^T$, $\bar{F}_0=1\otimes(\bar{f}_0^T+\bar{f}_{0e}^T)$, $\Lambda_1=[e_{x1}^T,e_{x2}^T,\ldots,e_{xN}^T]^T$, and $\Lambda_2=[e_{v1}^T,e_{v2}^T,\ldots,e_{vN}^T]^T$.

4.4. Finite-Time Leader-Following Consensus Control Analysis

In this part, we address a new algorithm for finite-time leader-following consensus of the nonlinear second order MASs. This algorithm is designed based on previous result especially Theorem 1. We construct the following sliding manifold such that on this surface, each agent $i\in\{0,1,\ldots,N\}$ interacts with its environment.

$$\begin{aligned}\varGamma_i=&\sum_{j=1}^{N}a_{ij}(v_i-v_j)+\alpha_{i0}(v_i-v_0)+\Upsilon_{1ii}\Big(\sum_{j=1}^{N}a_{ij}(x_i-x_j)\\ &+\alpha_{i0}(x_i-x_0)\Big)+\Upsilon_{2ii}\int_0^t\Big(\textstyle\sum_{j=1}^{N}a_{ij}(x_i-x_j)\\ &+\alpha_{i0}(x_i-x_0)\Big)d\tau-\Upsilon_{3ii}\displaystyle\int_0^t\bar{f}_{ie}^Td\tau.\end{aligned} \tag{5}$$

sliding mode manifold can be rewritten in the following compact form

$$\varGamma=\Lambda_2+\Upsilon_1\Lambda_1+\Upsilon_2\int_0^t\Lambda_1 d\tau-\Upsilon_3\int_0^t\bar{F}_E d\tau.$$

in which Υ_{1ii}, Υ_{2ii} and Υ_{3ii} are positive diagonal elements of diagonal matrices Υ_1, Υ_2 and Υ_3.

Remark 2: The finite-time leader-following consensus control objective of the MASs is that, for $i\in\{1.\cdots,N\}$, $x_i\rightarrow x_0$ and $v_i\rightarrow v_0$ as $t\rightarrow T^*$, and the trajectories x_i and v_i remain on the desired trajectories x_0 and v_0 for $t\geq T^*$, respectively. T^* is a positive real number which is dependent on initial values of the MASs. The finite-time stability of Systems (4) is equivalent to the finite-time leader-following consensus of the nonlinear second-order MASs (3).

4.5. Reachability of the Sliding Surface

Theorem 2: Assume that Assumptions 1 and 2 are satisfied. By employing distributed consensus protocol (6) for each agent i, $i\in\{0,1,\cdots,N\}$, if $[x_i,v_i]^T$ is a solution for MASs described with Equation (2) on $[t_0,\infty)$ in the sense of Filippov's solution then MASs reach consensus on the leader's states in finite-time input-to-state stability sense.

$$\bar{f}_i = \frac{1}{d_i + \alpha_{i0}} \Bigg[\sum_{j=1}^{N} a_{ij}\bar{f}_j + \alpha_{i0} f_0 - \Upsilon_{1ii} \Bigg(\sum_{j=1}^{N} a_{ij}(v_i - v_j) + \alpha_{i0}(v_i - v_0) \Bigg) - \Upsilon_{2ii} \Bigg(\sum_{j=1}^{N} a_{ij}(x_i - x_j) + \alpha_{i0}(x_i - x_0) \Bigg) + \Upsilon_{3ii}\bar{f}_{ie}^{T} - \vartheta(x_i, v_i, x_r, v_r)\nabla_i V(\Gamma) \Bigg], \tag{6}$$

in which $\nabla_i V(\Gamma) = sgn\big(\Gamma_i\big)$ denotes i'th element of vector $\nabla V(\Gamma) = sgn\big(\Gamma\big)$. Also, $\vartheta(x_i, v_i, x_r, v_r) = \wp + g_0$, where $\wp = \omega(\zeta_1\|\Lambda_1\| + \zeta_2\|\Lambda_2\|)$, $\omega = \|\tilde{L}\|\|\tilde{L}^{-1}\|$, and g_0 is a positive constant. Also, $\vartheta : R^{4n} \to R$ is a C^0 function.

Proof: Choose the Lyapunov candidate function as

$$V(\Gamma) = \Big\|\Gamma\Big\|_1 = \sum_{i=1}^{N} \Big\|\Gamma_i(t)\Big\| = \sum_{i=1}^{N} \max\Big(-\Gamma_i, \Gamma_i\Big). \tag{7}$$

then if $\dot{V}$ exists, one computes $\dot{V}$ and chooses appropriate ϑ such that $\dot{V}$ is bounded below zero (i.e. $\dot{V}(t) \le -\varepsilon$ for all $t \ge t_0$).

Remark 3: The set of points where the Lyapunov function is not (continuously) differentiable has measure zero. These points can not be ignored, and then the smooth Lyapunov theory is not applicable. In other words, systems' trajectories might tend to infinity in the nonzero length of time along these discontinuities ([2]).

To proceed, the authors used non-smooth Lyapunov theory as developed in [3]. From Equation (2) and since $[x_i, v_i]^T$ is a solution to MASs on $[t_0, \infty)$, we obtained that the following equation holds a.e. in $[t_0, \infty)$

$$\begin{bmatrix} \bar{F}_E \\ \Lambda_1 \\ \dot{\Lambda}_1 \\ \dot{\Lambda}_2 \end{bmatrix} \in \mathrm{K} \begin{bmatrix} \bar{F}_E \\ \Lambda_1 \\ \dot{\Lambda}_1 \\ (\tilde{L} \otimes I_m)\Big(\bar{H} - \bar{H}_0 + \bar{F} - \bar{F}_0\Big) \end{bmatrix}$$

for the derivative of the sliding surface, we can obtain the following equation with calculus derived in [2, Theorem 1].

$$\dot{\Gamma} \in \mathrm{K}\Big[(\tilde{L}\otimes I_m)(\bar{H} - \bar{H}_0 + \bar{F} - \bar{F}_0) - \Upsilon_3\bar{F}_E\Big] + \Upsilon_1\dot{\Lambda}_1 + \Upsilon_2\Lambda_1$$

The controller (6) can be rewritten in the following collective form

$$\bar{F} = (\tilde{L}\otimes I_m)^{-1}\Big(\bar{F}_0 - \Upsilon_1\Lambda_2 - \Upsilon_2\Lambda_1 + \Upsilon_3\bar{F}_E - \vartheta\nabla V(\Gamma)\Big). \tag{8}$$

From (8) we obtain

$$\dot{\Gamma} \in \mathrm{K}\Big[(\tilde{L}\otimes I_m)(\bar{H} - \bar{H}_0 - \vartheta\nabla V(\Gamma))\Big]. \tag{9}$$

since the trajectory $[x_i, v_i]$ is absolutely continuous on compact intervals and the trajectory $[x_0, v_0]^T$ is C^1 function on $[t_0, \infty)$, one can conclude that Γ is absolutely continuous on compact intervals. Absolute continuity of Γ implies that V is absolutely continuous on compact intervals.

$$V(\Gamma) = \Big\|\Gamma\Big\|_1 = \sum_{i=1}^{N} \max\Big(-\Gamma_i, \Gamma_i\Big).$$

Since V is a max function, with Lemma 2, one has

$$\mathscr{L}_{\mathscr{K}}V = \bigcap_{\xi\in\partial V} \xi^T \mathrm{K}\Big[(\tilde{L}\otimes I_m)(\bar{H} - \bar{H}_0 - \vartheta\nabla V(\Gamma))\Big],$$

other form of the above equation can be written by Lemma 1 such as

$$\dot{V} = \xi^T\dot{\Gamma} \quad a.e. \quad \forall\xi \in \partial V(\Gamma), \tag{10}$$

from (9) , (10), $\forall\xi \in \partial V(\Gamma)$, some $\beta \in \partial V(\Gamma)$ and some $\gamma \in \mathrm{K}\big[\bar{H} - \bar{H}_0\big]$, one has

$$\dot{V}(t) = -\xi^T(\tilde{L}\otimes I_m)\vartheta\cdot\beta + \xi^T\big[(\tilde{L}\otimes I_m)\gamma\big].$$

We choose $\xi = \arg\min\{\|\Omega\||\Omega \in \partial V(\Gamma(t))\}$ then from convexity $\partial V(\Gamma(t))$, one can conclude

$$\dot{V}(t) \le -\xi^T(\tilde{L}\otimes I_m)\vartheta\xi + \xi^T(\tilde{L}\otimes I_m)\gamma,\ a.e. \tag{11}$$

where ϑ is given by

$$\vartheta = \sigma_{max}(\tilde{L}\otimes I_m)\frac{\Big\|\bar{H} - \bar{H}_0\Big\|}{\sigma_{min}(\tilde{L}\otimes I_m)} + \sigma_{max}(\tilde{L}\otimes I_m)\delta_0, \tag{12}$$

such that δ_0 is non-negative constant. From (11) and (12), we conclude

$$\dot{V} \le -\vartheta\|\xi\|^2\sigma_{min}(\tilde{L}\otimes I_m) + \|\xi\|\Big(\sigma_{min}(\tilde{L}\otimes I_m)\Big\|\bar{H} - \bar{H}_0\Big\|\Big),\ a.e.$$

where $\Big\|\bar{H} - \bar{H}_0\Big\| = \sup\Big\{\|\gamma\||\gamma \in \mathrm{K}\Big[\bar{H} - \bar{H}_0\Big]\Big\}$, $\sigma_{\min}(\mathscr{A})$, and $\sigma_{\max}(\mathscr{A})$ denote minimum and maximum singular values of $\mathscr{A}$, respectively. The Equation (12) yields

$$\dot{V}(t) \le \Big(\|\xi\| - \|\xi\|^2\Big)\sigma_{max}(\tilde{L}\otimes I_m)\Big\|\bar{H} - \bar{H}_0\Big\|, \quad a.e. \text{ on } [t_0, \infty). \tag{13}$$

Calculus of generalized gradients yield $\partial V(0) = [-1, 1]^n$, from convexity of the $\partial V(\Gamma)$ and V, $\partial V(\Gamma)\bigcap(-1, 1)^n = \{\}$, $\forall\Gamma \ne 0$, [2, Proposition 2.2.9]. Thus $\|\xi\| \ge 1$, $\forall\Gamma(t) \ne 0$ and from (13), one can conclude $\dot{V} \le -\varepsilon$, $\forall\Gamma \ne 0$ $a.e.$ on $[t_0, \infty)$. Since $V(t)$ is absolutely continuous, one can conclude that $\exists T \in [t_0, \infty)$ such that $\Gamma = 0$, $\forall t \ge T$ by Theorem 1. Now, existence, uniqueness and continuation of a Filippov's solution for (4) should be proved. This follows from the [2, Theorem 5](skipped for brevity).

4.6. Sliding Motion Analysis

Theorem 2 ensures that the terminal sliding mode surfaces $\Gamma = 0$ can be reached in a finite time. From now on, we show the input-to-state stability of the states is obtained in a finite time. In other words, based on Theorem 1, there exists a $T^* \in [t_0, \infty)$ such that $\forall t \geq T^*$, the errors of position and velocity tracking are bounded on the sliding phase according to infinity norm of external forces.

Consider the following Lyapunov function

$$V_* = 0.5\Lambda_1^T \Lambda_1,$$
$$\dot{V}_* = \bigcap_{\mu \in \partial V_*} \mu^T \mathrm{K}\Big[-\Upsilon_1\Lambda_2 - \Upsilon_2\Lambda_1 + \Upsilon_3\bar{F}_E\Big].$$

Since V_* is smooth and the solution $[x_i, v_i]^T$ is absolutely continuous on compact intervals and the desired trajectory$[x_0, v_0]^T$ is C^1 function on $[t_0, \infty)$, one has that V_* is absolutely continuous on compact intervals. Therefore,

$$\dot{V}_* = \bigcap_{\mu \in \partial V_*} \mu^T \mathrm{K}\Big[-\Upsilon_1\Lambda_2 - \Upsilon_2\Lambda_1 + \Upsilon_3\bar{F}_E\Big]$$
$$\dot{V}_* = \nabla V^T \mathrm{K}\Big[\Upsilon_3\bar{F}_E - \Upsilon_1\Lambda_2 - \Upsilon_2\Lambda_1\Big]$$
$$\subset \Big[\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big]^T\Big[\Upsilon_3\bar{F}_E - \Upsilon_1\Lambda_2 - \Upsilon_2\Lambda_1\Big]$$
$$= -(1-\Theta)\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big)^T\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big)$$
$$-\Theta\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big)^T\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big)$$
$$+\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big)^T\Big(\Upsilon_3\bar{F}_E\Big)$$
$$\leq -(1-\Theta)\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big)^T\Big(\Upsilon_1\Lambda_2 + \Upsilon_2\Lambda_1\Big) < 0.$$

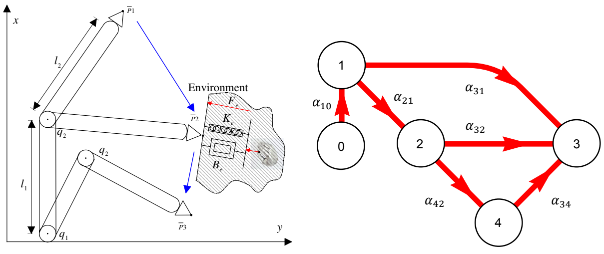
provided that $0 < \Theta < 1$ and $\forall\|\Lambda_1\| \geq \chi(\bar{F}_E, \Theta)$. By Theorem 1, FTISS of the MASs is proved.

Remark 4: It should be noted that on the sliding surfaces Γ, one can obtained $\|\Lambda_2\|$ in the terms of $\|\Lambda_1\|$, and this means that Function $\chi(\cdot)$ only depends on $\bar{F}_E$ and Θ. ■

5. SIMULATION

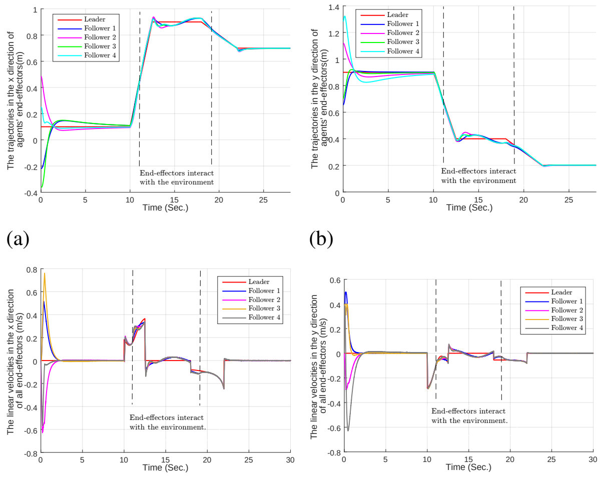
In this section, some simulations are performed with a 2-DOF planar robot manipulator([38]) to evaluate the effectiveness of the proposed method. We consider five Lagrangian MASs which are tagged $0, 1, \cdots, 4$. In addition, we suppose the leader to be agent 0 and the other agents are the followers. Fig. 1(a) shows two-degrees-of-freedom manipulator. Also, the communication topology depicted in Fig. 1(b). Let mass of each link $m_1 = m_2 = 0.2Kg$, the lengths $l_1 = l_1 = 0.5m$, and the acceleration of gravity 9.8.

In this simulation, the manipulators are commanded to move from the point $\bar{P}1 = [0.1, 0.9]$ to the point $\bar{P}2 = [0.9, 0.4]$. Afterwards, the manipulator follows the desired trajectory between the points $\bar{P}2$ to $\bar{P}3 = [0.7, 0.2]$. In the meanwhile, the manipulators interact with the considered environment while the force $\bar{F}_E$ is acting on the



(a) (b)

Fig. 1: (a): 2-DOF manipulator and Environment. (b): Communication topology of the agents.



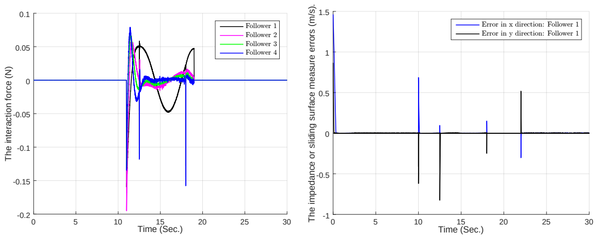
(a) (b) (c) (d)

Fig. 2: (a): The trajectories in the x direction. (b): The trajectories in the y direction. (c): The linear velocities in the x direction. (d): The linear velocities in the y direction.

end-effectors (constrained motion) between the time $t = 11s$ to $t = 19s$. The interaction force with the environment is described as $\bar{F}_E = -K_e(x - x_0) - B_e(\dot{x} - \dot{x}_0) + 0.1 \times \sin(t) \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where x_0 and $\dot{x}_0$ are desired Cartesian trajectory and velocity and $K_e = 5 \times I_2$ and $B_e = I_2$. According to Assumption 1, it should be noted that $\bar{F}_E$ is locally bounded. The desired trajectory is first-order polynomials interpolated in the task space among the points $\bar{P}1$, $\bar{P}2$, and $\bar{P}3$ with zero velocities and accelerations at those points.

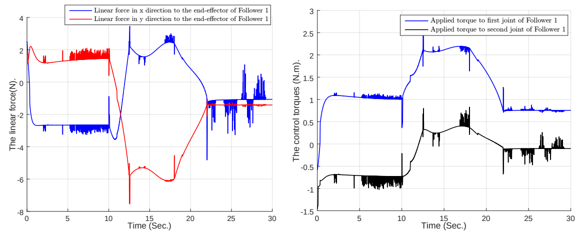
In this simulation, the manipulators are impedance controlled to regulate the interaction force when the arm moves through the environment and also to follow the desired path closely in both free and constrained motions. The the desired sliding surface is in the form of (5) in which $\Upsilon_{1ii} = M_d^{-1}C_d$, $\Upsilon_{2ii} = M_d^{-1}K_d$, and $\Upsilon_{3ii} = M_d^{-1}$ with the following values $M_d = 0.5 \times I_2$, $K_d = 0.25 \times I_2$, and $C_d = 1.2 \times I_2$.

Fig. 2 shows end-effector's Cartesian positions and linear velocities in x and y directions. In other words, MASs achieve consensus on desired trajectories. The initial positions and velocities of the joints are chosen randomly.



(a) (b)

Fig. 3: (a): The environment or interaction force (b): The impedance measure errors for the manipulator 1.



(a) (b)

Fig. 4: (a): The control linear forces applied to the end-effector of the manipulator 1. (b): The control torques applied to the joints of the manipulator 1.

The interaction forces f_{ie}, $\forall i \in \{1, \cdots, 4\}$ are shown in Fig. 3(a). This force before and after the contact is zero. During $t = 11s$ untill $t = 19s$, a collision force appeared. After the collision, the interaction force f_{ie} was back to the values determined by the desired impedance or sliding surface.

The impedance surface, which measures errors in x and y axes for the manipulator 1, are plotted in Fig. 3(b). When the manipulator is in free motion, both of the errors in x and y axes are small while they are noticeable at the moment of collision.

Finally, the control linear force $f_1 \in R^2$ according to the Equation (6) in both x and y directions for the manipulator 1 are shown in Fig. 4(a) which are applied on the end-effector with the parameters mentioned earlier. In addition, the control torques applied to the joints are depicted in Fig. 4(b).

6. CONCLUSION

In this work, the focus of attention was on the finite time input-to-state stability of DDSs. We addressed a theorem for assurance of FTISS of discontinuous dynamical systems, and also this method is used for MASs, which are modeled as Euler-Lagrange dynamics. The introduced sliding surfaces represent the relationship between the position in Cartesian space and the interaction/external forces, which reflect the error between the states' trajectories and the surface. The desired impedance is achieved when the impedance measure error is zero, which means the states' trajectories reach and remain on the desired sliding surfaces, showing that the dynamical behavior of the system in the interaction port with the environment. The stability of the systems is preserved while external forces are acting on the end-effectors, and this demonstrates the robustness of the systems. Also, simulation results are consistent with the proposed method, which shows the efficiency of the algorithm. Switching communication topologies among the agents, derivation of the stability time, and guaranteeing the existence of an extended Filippov's solution for the system can be considered as future work.

REFERENCES

[1] J. Cortes, "Discontinuous dynamical systems," *Control Systems, IEEE*, vol. 28, no. 3, pp. 36–73, 2008.

[2] B. Paden and S. Sastry, "A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Transactions on Circuits and Systems*, vol. 34, no. 1, pp. 73–82, 1987.

[3] D. Shevitz, B. Paden, *et al.*, "Lyapunov stability theory of nonsmooth systems," *IEEE Transactions on automatic control*, vol. 39, no. 9, pp. 1910–1914, 1994.

[4] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth analysis and control theory*, vol. 178. Springer Science & Business Media, 2008.

[5] J.-P. Aubin and H. Frankowska, *Set-valued analysis*. Springer Science & Business Media, 2009.

[6] E. D. Sontag, "Smooth stabilization implies coprime factorization," *Automatic Control, IEEE Transactions on*, vol. 34, no. 4, pp. 435–443, 1989.

[7] E. D. Sontag, "Some connections between stabilization and factorization," in *Proc. of the 28th IEEE Conference on Decision and Control (CDC 1989)*, vol. 1, p. 3, 1989.

[8] E. D. Sontag, "Further facts about input to state stabilization," *IEEE Transactions on Automatic Control*, vol. 35, no. 4, pp. 473–476, 1990.

[9] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for iss systems and applications," *Mathematics of Control, Signals and Systems*, vol. 7, no. 2, pp. 95–120, 1994.

[10] Z.-P. Jiang, I. M. Mareels, and Y. Wang, "A lyapunov formulation of the nonlinear small-gain theorem for interconnected iss systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996.

[11] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse lyapunov theorem for robust stability," *SIAM Journal on Control and Optimization*, vol. 34, no. 1, pp. 124–160, 1996.

[12] E. D. Sontag, "The iss philosophy as a unifying framework for stability-like behavior," in *Nonlinear control in the year 2000 volume 2*, pp. 443–467, Springer, 2001.

[13] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351–359, 1995.

[14] V. T. Haimo, "Finite time controllers," *SIAM Jour-*

nal on Control and Optimization, vol. 24, no. 4, pp. 760–770, 1986.

[15] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, no. 2, pp. 101–127, 2005.

[16] E. Moulay and W. Perruquetti, "Finite time stability of nonlinear systems," in *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*, vol. 4, pp. 3641–3646, IEEE, 2003.

[17] Y. Orlov, "Finite time stability and quasihomogeneous control synthesis of uncertain switched systems with application to underactuated manipulators," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pp. 4566–4571, IEEE, 2005.

[18] Y. Hong, Z.-P. Jiang, and G. Feng, "Finite-time input-to-state stability and applications to finite-time control design," *SIAM Journal on Control and Optimization*, vol. 48, no. 7, pp. 4395–4418, 2010.

[19] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 4, pp. 461–474, 1998.

[20] R. A. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[21] M. Johansson, A. Rantzer, *et al.*, "Computation of piecewise quadratic lyapunov functions for hybrid systems," *IEEE transactions on automatic control*, vol. 43, no. 4, pp. 555–559, 1998.

[22] S. Pettersson and B. Lennartson, "Lmi for stability and robustness of hybrid systems," in *American Control Conference, 1997. Proceedings of the 1997*, vol. 3, pp. 1714–1718, IEEE, 1997.

[23] G. Ferrari-Trecate, F. A. Cuzzola, D. Mignone, and M. Morari, "Analysis of discrete-time piecewise affine and hybrid systems," *Automatica*, vol. 38, no. 12, pp. 2139–2146, 2002.

[24] C. Cai and A. R. Teel, "Results on input-to-state stability for hybrid systems," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pp. 5403–5408, IEEE, 2005.

[25] L. Vu, D. Chatterjee, and D. Liberzon, "Iss of switched systems and applications to switching adaptive control," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pp. 120–125, IEEE, 2005.

[26] D. Liberzon and D. Nešić, "Stability analysis of hybrid systems via small-gain theorems," in *Hybrid systems: computation and control*, pp. 421–435, Springer, 2006.

[27] W. Heemels and S. Weiland, "Input-to-state stability and interconnections of discontinuous dynamical systems," *Automatica*, vol. 44, no. 12, pp. 3079–3086, 2008.

[28] M. Xiaowu, G. Yang, and Z. Wei, "Integral input-to-state stability for one class of discontinuous dynamical systems," in *Proceedings of the 29th Chinese Control Conference*, 2010.

[29] E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti, "Robustness of finite-time stability property for sliding modes," in *Joint SSSC, TDS, FDA 2013*, 2013.

[30] J. A. Moreno and M. Osorio, "Strict lyapunov functions for the super-twisting algorithm," *Automatic Control, IEEE Transactions on*, vol. 57, no. 4, pp. 1035–1040, 2012.

[31] A. Filippov, *Differential Equations with Discontinuous Righthand Sides: Control Systems*, vol. 18. Springer Science & Business Media, 1988.

[32] T. Liu, Z.-P. Jiang, and D. J. Hill, *Nonlinear control of dynamic networks*. CRC Press, 2014.

[33] W. Rudin, *Principles of mathematical analysis*, vol. 3. McGraw-Hill New York, 1964.

[34] L. F. Richardson, *Measure and integration: a concise introduction to real analysis*. John Wiley & Sons, 2009.

[35] G. B. Folland, *Real analysis: modern techniques and their applications*. John Wiley & Sons, 2013.

[36] G. Wen, Y. Yu, Z. Peng, and A. Rahmani, "Consensus tracking for second-order nonlinear multi-agent systems with switching topologies and a time-varying reference state," *International Journal of Control*, vol. 89, no. 10, pp. 2096–2106, 2016.

[37] G. Wen, Y. Yu, Z. Peng, and A. Rahmani, "Distributed finite-time consensus tracking for nonlinear multi-agent systems with a time-varying reference state," *International Journal of Systems Science*, vol. 47, no. 8, pp. 1856–1867, 2016.

[38] J. J. Craig, *Introduction to robotics: mechanics and control*, vol. 3. Pearson/Prentice Hall Upper Saddle River, NJ, USA:, 2005.